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TECHNICAL NOTE

The analogy between fluid friction and heat transfer of laminar mixed convection on flat plates

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1. INTRODUCTION

THE ANALOGY between fluid friction and heat or mass transfer was introduced by Reynolds [1] in 1874 for fluid of $Pr = 1$. Some modified analogies for $Pr > 1$ have also been developed by Prandtl [2], Karman [3], and Colburn [4]. These analogies have been applied successfully to laminar and turbulent forced convection flows over a flat plate and in pipes. If one of the friction coefficient, the heat transfer Nusselt number, and the mass transfer Sherwood number is known, the other two are readily obtainable through these analogies.

Although these analogies are very useful for forced convection of $Pr > 1$, they are not valid for the case of $Pr < 1$. For the sake of completeness, we have recently developed an analogy for small Prandtl numbers [5]. We have also proposed analogies [6] for the laminar natural convection on vertical and horizontal flat plates with uniform wall temperature or heat flux.

Now, it is the time to ask the question: whether the analogy for the mixed convection exists? If it does exist, what would be the form of the analogy? This work intends to find the answer.

2. FORMULATIONS

In the analysis of laminar mixed convection heat transfer from vertical and horizontal flat plates with uniform wall temperature (UWT) and uniform wall heat flux (UHF), an appropriate mixed convection parameter has been introduced [7–10] as

$$\xi = [1+(\omega Re)^{1/2}/(\sigma Ra)^{1/n}]^{-1} \quad (1)$$

where

$$\omega = Pr/(1+Pr)^{1/3}; \quad \sigma = Pr/(1+Pr) \quad (2)$$

and

$$Re = u_\infty x/\nu; \quad Ra = g\beta T^* x^3/\alpha\nu \quad (3)$$

are, respectively, the Reynolds number and the Rayleigh number with the characteristic temperature $T^* = T_w - T_\infty$ for the UWT case and $T^* = q_w x/k$ for the UHF case. For a vertical plate, $n = 4$ for the UWT case and $n = 5$ for the UHF case. For a horizontal plate, $n = 5$ and 6 for the UWT and UHF cases, respectively. The parameter ξ describes the relative strength of the buoyancy force to the inertia force. For the case of pure forced convection, $\xi = 0$, while, for the case of pure free convection, $\xi = 1$.

Very precise numerical solutions of laminar mixed convection from various flat plates have been reported [7–9] over the entire mixed convection region ($0 \leqslant \xi \leqslant 1$) for any Prandtl number between 0.001 and infinity. The wall shear stress can be calculated from these numerical results by the relation

$$\tau_w = \rho\nu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \rho(\alpha\nu/x^2)\lambda^3 f''(\xi,0) \quad (4)$$

where

$$\lambda = (\omega Re)^{1/2} + (\sigma Ra)^{1/n} \quad (5)$$

and $f''(\xi,0)$ is the second derivative of the dimensionless stream function $f = \psi/\alpha\lambda$ at the wall. The prime denotes partial differentiation with respect to the dimensionless coordinate $\eta = (y/x)\lambda$. In addition, for the case of uniform wall temperature the local Nusselt number can be obtained from

$$Nu = -\lambda\theta'(\xi,0) \quad (6)$$

in which $\theta'(\xi,0)$ is the gradient of the dimensionless temperature $\theta(\xi,\eta) = (T-T_w)/(T_w-T_\infty)$ at the wall, while for uniform heating

$$Nu = \lambda/\phi(\xi,0) \quad (7)$$

where the dimensionless wall temperature $\phi(\xi,0) = (T_w - T_\infty)\lambda/(q_w x/k)$.

Dividing equation (4) by equation (6) or (7) gives

$$\tau_w/Nu = \rho(\alpha\nu/x^2)\lambda^2\Gamma(\xi,0) \quad (8)$$

where

$$\Gamma(\xi,0) = -\frac{f''(\xi,0)}{\theta'(\xi,0)} \quad \text{or} \quad \Gamma(\xi,0) = f''(\xi,0)\phi(\xi,0). \quad (9)$$

Furthermore, by using the following relation between λ and ξ

$$\lambda = (\omega Re)^{1/2}/(1-\xi) = (\sigma Ra)^{1/n}/\xi \quad (10)$$

and the definitions of the friction coefficients

$$C_f = 2\tau_w/\rho u_\infty^2; \quad C_N = \tau_w/(\rho\alpha\nu/x^2) \quad (11)$$

equation (8) becomes

$$\frac{C_f/2}{Nu/Re} = (1-\xi)^{-2}(1+Pr)^{-1/3}\Gamma(\xi,0) \quad (12)$$

or

$$\frac{C_N}{NuRa^{2/n}} = \xi^{-2}\left(\frac{Pr}{1+Pr}\right)^{2/n}\Gamma(\xi,0). \quad (13)$$

It is our purpose to find a function of Prandtl number, $P(Pr)$, and a function of mixed convection parameter, $X(\xi)$, such that for each of the four mixed convection cases the numerical results of

$$\frac{C_f/2}{Nu/Re}P(Pr)X(\xi) = \frac{X(\xi)}{(1-\xi)^2}\frac{P(Pr)}{(1+Pr)^{1/3}}\Gamma(\xi,0) \quad (14)$$

or

NOMENCLATURE

a, b, c	constants
C_f	local friction coefficient, $2\tau_w/\rho u_\infty^2$
C_N	local friction coefficient, $\tau_w/(\rho\alpha v/x^2)$
f	dimensionless stream function, $\psi/\alpha\lambda$
g	gravitational acceleration [m s^{-2}]
h	local heat transfer coefficient [J s^{-1} m^{-1} K^{-1}]
k	thermal conductivity of fluid [J s^{-1} m^{-1} K^{-1}]
Nu	local Nusselt number, hx/k
P	function of Pr
Pr	Prandtl number, ν/α
q_w	wall heat flux [J s^{-1} m^{-2}]
Ra	Rayleigh number, $g\beta T^* x^3/\alpha\nu$
Re	Reynolds number, $u_\infty x/\nu$
T	fluid temperature [K]
T^*	characteristic temperature, $T_w - T_\infty$ for UWT; and $q_w x/k$ for UHF [K]
u	velocity component in x direction [m s^{-1}]
x, y	coordinates parallel and normal to the plate [m]
X	function of ξ.

Greek symbols

α	thermal diffusivity [m^2 s^{-1}]
β	thermal expansion coefficient [K^{-1}]
$\Gamma(\xi, 0)$	$-f''(\xi,0)/\theta'(\xi,0)$ for UWT; and $f''(\xi,0)\phi(\xi,0)$ for UHF
η	dimensionless coordinate, $(y/x)\lambda$
θ	dimensionless temperature, $(T-T_\infty)/(T_w-T_\infty)$
λ	$(\omega Re)^{1/2} + (\sigma Ra)^{1/n}$
ν	kinematic viscosity [m^2 s^{-1}]
ξ	mixed convection parameter, $[1+(\omega Re)^{1/2}/(\sigma Ra)^{1/n}]^{-1}$
ρ	density [kg m^{-3}]
σ	$Pr/(1+Pr)$
τ_w	wall shear stress, $\rho\nu(\partial u/\partial y)_{y=0}$ [kg m^{-1} s^{-2}]
ϕ	dimensionless temperature, $(T-T_\infty)\lambda/(q_w x/k)$
ω	$Pr/(1+Pr)^{1/3}$.

Subscripts

w	adjacent to the wall
∞	far from the wall.

$$\frac{C_N}{NuRa^{2/n}}P(Pr)X(\xi) = \frac{X(\xi)}{\xi^2}P(Pr)\left(\frac{Pr}{1+Pr}\right)^{2/n}\Gamma(\xi,0) \quad (15)$$

are close to a constant over a wide range of Pr and ξ.

3. ANALOGIES FOR THE FORCED CONVECTION DOMINATED REGION

At the forced convection dominated region of $0 \leqslant \xi \leqslant 0.2$ and for $0.7 \leqslant Pr \leqslant \infty$, the functions $P(Pr) = Pr^{1/3}$ and $X(\xi) = 1$ are appropriate for both the UWT and UHF cases. These functions and the numerical data of $f''(\xi,0)$ and $\theta'(\xi,0)$ or $\phi(\xi,0)$ were substituted into equation (14), and the results have been presented in Table 1. It is found from this table that the numerical results of

$$\frac{C_f/2}{Nu/RePr^{1/3}} = (1-\xi)^{-2}\left(\frac{Pr}{1+Pr}\right)^{1/3}\Gamma(\xi,0) \quad (16)$$

are close to a constant, i.e. 0.98 ~ 1.02 for UWT; and 0.716 ~ 0.73 for UHF. As a result, in the forced convection dominated region ($0 \leqslant \xi \leqslant 0.2$) the analogy can be stated as

$$\frac{C_f/2}{Nu/RePr^{1/3}} = 1 \quad (17)$$

for the case of uniform wall temperature; and

$$\frac{C_f/2}{Nu/RePr^{1/3}} = 0.72 \quad (18)$$

for the case of uniform heating. These analogies are valid over the range of $0.7 \leqslant Pr \leqslant \infty$. Note that equation (17) is exactly the Colburn analogy [4].

4. ANALOGY FOR THE NATURAL CONVECTION DOMINATED REGION

When $n = 4$ and the numerical data of $f''(\xi,0)$ and $\theta'(\xi,0)$ for laminar mixed convection from an isothermal vertical plate [7] have been substituted into equation (13), it is very interesting to find that the numerical results of

$$\frac{C_N/2}{NuRa^{1/2}} = \frac{1}{2\xi^2}\left(\frac{Pr}{1+Pr}\right)^{1/2}\Gamma(\xi,0) \quad (19)$$

converge to a constant about 1.15 for any ξ and Pr in the region of $0.6 \leqslant \xi \leqslant 1$ and $0.7 \leqslant Pr \leqslant \infty$, as has been shown in Table 2.

It is more interesting to find that, in the region of $0.6 \leqslant \xi \leqslant 1$ and $0.7 \leqslant Pr \leqslant \infty$, the numerical results of

$$\frac{C_N/2}{NuRa^{2/n}}\left(\frac{Pr}{1+Pr}\right)^{(n-4)/2n} = \frac{1}{2\xi^2}\left(\frac{Pr}{1+Pr}\right)^{1/2}\Gamma(\xi,0) \quad (20)$$

Table 1. Numerical results of $(C_f/2)/(Nu/RePr^{1/3})$ in the forced convection dominated region

ξ	$Pr = 0.7$	1	10	100	1000	10 000
UWT case:						
0	1.007	1.000	0.9847	0.9806	0.9805	0.9805
0.1	1.008	1.001	0.9877	0.9813	0.9812	0.9812
0.2	1.021	1.013	1.006	0.9934	0.9942	0.9943
UHF case:						
0	0.7264	0.7235	0.7185	0.7162	0.7161	0.7161
0.1	0.7267	0.7237	0.7204	0.7164	0.7164	0.7164
0.2	0.7306	0.7278	0.7293	0.7212	0.7212	0.7212

Table 2. Numerical results of $(C_N/2NuRa^{2/n})[Pr/(1+Pr)]^{(n-4)/2n}$ in the natural convection dominated region

ξ	$Pr = 0.7$	1	10	100	1000	10000
Vertical plate, UWT case ($n = 4$):						
0.6	1.122	1.157	1.196	1.213	1.220	1.222
0.7	1.138	1.139	1.155	1.171	1.178	1.180
0.8	1.136	1.134	1.140	1.155	1.162	1.165
0.9	1.137	1.133	1.135	1.150	1.158	1.161
1	1.137	1.133	1.134	1.149	1.157	1.160
Vertical plate, UHF case ($n = 5$):						
0.6	1.111	1.127	1.189	1.208	1.213	1.214
0.7	1.158	1.168	1.206	1.217	1.220	1.220
0.8	1.183	1.189	1.215	1.221	1.222	1.223
0.9	1.194	1.199	1.218	1.222	1.223	1.224
1	1.197	1.201	1.218	1.222	1.223	1.224
Horizontal plate, UWT case ($n = 5$):						
0.6	1.022	1.032	1.087	1.100	1.099	1.106
0.7	1.022	1.028	1.062	1.074	1.078	1.079
0.8	1.027	1.031	1.055	1.066	1.070	1.071
0.9	1.030	1.032	1.053	1.063	1.068	1.069
1	1.030	1.033	1.052	1.063	1.067	1.069
Horizontal plate, UHF-case ($n = 6$):						
0.6	0.964	0.967	1.045	1.050	1.052	1.064
0.7	0.990	1.003	1.047	1.047	1.047	1.043
0.8	1.003	1.013	1.049	1.047	1.046	1.046
0.9	1.009	1.018	1.049	1.047	1.045	1.045
1	1.010	1.018	1.049	1.047	1.045	1.045

approach to a constant for each of the other three mixed convection cases. Consequently, in the natural convection dominated region, $P(Pr) = [Pr/(1+Pr)]^{(n-4)/2n}$ and $X(\xi) = 1$ for the four mixed convection cases.

In the natural convection dominated region ($0.6 \leqslant \xi \leqslant 1$) and for moderate and large Prandtl numbers ($0.7 \leqslant Pr \leqslant \infty$), the analogies between fluid friction and heat transfer of the four convection cases can be expressed in a general form as

$$\frac{C_N/2}{NuRa^{2/n}}\left(\frac{Pr}{1+Pr}\right)^{(n-4)/2n} = c \tag{21}$$

where the constant $c = 1.15$ and 1.20 for the cases of a vertical plate with uniform wall temperature and heat flux, respectively. For laminar mixed convection over a horizontal plate, $c = 1.05$ for the UWT case and 1.02 for the UHF case.

5. ANALOGY FOR THE TRUE MIXED CONVECTION REGION

In this section, we try to find the analogies for the four mixed convection cases in the region of $0.2 \leqslant \xi \leqslant 0.6$ where the buoyancy and inertia forces are comparable in magnitude. By multiplying equation (12) with

$$P(Pr) = Pr^{1/3}\left(\frac{Pr}{1+Pr}\right)^{1/24} \tag{22}$$

we found that, for each of the four convection cases, the numerical results of

$$\frac{C_f/2}{Nu/RePr^{1/3}}\left(\frac{Pr}{1+Pr}\right)^{1/24} = (1-\xi)^{-2}\left(\frac{Pr}{1+Pr}\right)^{3/8}\Gamma(\xi,0) \tag{23}$$

are nearly a constant over the range of $0.7 \leqslant Pr \leqslant \infty$ for a specified value of ξ between 0.2 and 0.6. These numerical results are almost independent of Pr and can be regarded as a function of ξ.

To correlate the numerical data of equation (23) over the range of $0.2 \leqslant \xi \leqslant 0.6$, we chose, after many trials, the form of

$$\frac{C_f/2}{Nu/RePr^{1/3}}\left(\frac{Pr}{1+Pr}\right)^{1/24} = a+b\xi^6. \tag{24}$$

The constants a and b for the four mixed convection cases have been determined and presented in Table 3. Comparison of equation (24) with the numerical data of mixed convection on vertical and horizontal plates has been made in Figs. 1 and 2, respectively. These figures indicate fairly good agreement. Over the entire region of $0.2 \leqslant \xi \leqslant 0.6$ and $0.7 \leqslant Pr \leqslant \infty$, the maximum deviations of equation (24) from the numerical data are less than 3 and 4.5% for vertical plate with UWT and UHF, respectively. The maximum errors for horizontal plate with UWT and UHF are 2.8 and 4%, respectively.

Table 3. Values of a and b in the analogy equation (24) for the four mixed convection cases

Cases	a	b
Vertical plate:		
UWT case	1	103
UHF case	0.7	106
Horizontal plate:		
UWT case	0.96	88
UHF case	0.69	88

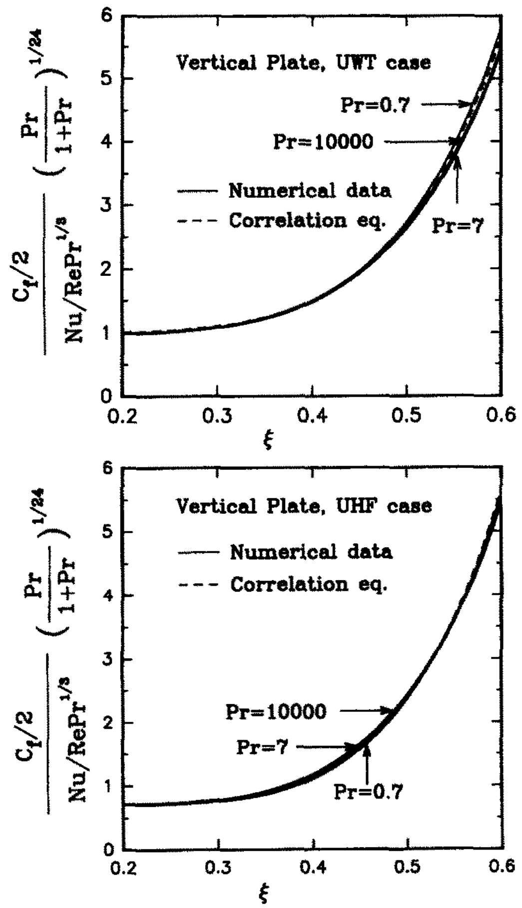

FIG. 1. Comparison between the correlation equation and the numerical data of $[C_f/(2Nu/RePr^{1/3})][Pr/(1+Pr)]^{1/24}$ for a vertical plate with (a) uniform wall temperature; (b) uniform heat flux.

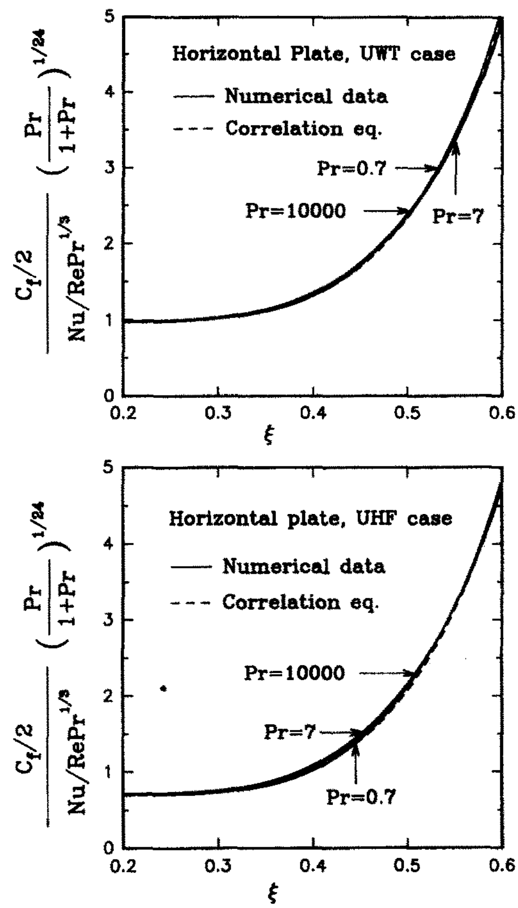

FIG. 2. Comparison between the correlation equation and the numerical data of $[C_f(2Nu/RePr^{1/3})][Pr/(1+Pr)]^{1/24}$ for a horizontal plate with (a) uniform wall temperature; (b) uniform heat flux.

It is worth to note that equation (24) can be reduced easily to equation (17) or (18) for $\xi \leqslant 0.2$ and $0.7 \leqslant Pr \leqslant \infty$. Equation (24) can also be reduced to equation (21) for $\xi \geqslant 0.6$ via a relation between $C_f/(Nu/Re)$ and $C_N/(NuRa^{2/n})$, which is obtainable from equations (14) and (15).

6. CONCLUSIONS

The analogies between heat transfer and fluid friction of laminar mixed convection on vertical and horizontal flat plates with uniform wall temerature and heat flux have been developed in this paper for moderate and large Prandtl numbers ($0.7 \leqslant Pr \leqslant \infty$). Three different analogies have been proposed for the forced convection dominated region ($0 \leqslant \xi \leqslant 0.2$), the natural convection dominated region ($0.6 \leqslant \xi \leqslant 1$), and the true mixed convection region ($0.2 \leqslant \xi \leqslant 0.6$) as equations (17), (21), and (24), respectively.

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REFERENCES

1. O. Reynolds, On the extent and action of the heating surface for steam boilers, *Proc. Manchester Lit. Phil. Soc.* **14,** 7–12 (1874).
2. L. Prandtl, Eine beziehung zwischen warmeaustausch und der flussifkeiten, *Phyz. Z.* **11,** 1072–1078 (1910).
3. T. von Karman, The analogy between fluid friction and heat transfer, *Trans. ASME* **61,** 705–710 (1939).
4. A. P. Colburn, A method of correlating forced convection heat transfer data and a comparison with fluid friction, *Trans. Am. Inst. Chem. Engrs* **29,** 174–210 (1933).
5. H.-T. Lin, The analogy between fluid friction and heat transfer of laminar forced convection on a flat plate, *Wärme- und Stoffübertragung* **29,** 181–184 (1994).
6. H.-T. Lin, The analogy between fluid friction and heat transfer of laminar natural convection on vertical and horizontal flat plates, *Int. J. Heat Mass Transfer* **35,** 1325–1327 (1992).
7. H.-T. Lin and C.-C. Chen, Mixed convection on vertical plate for fluids of any Prandtl number, *Wärme- und Stoffübertragung* **22,** 159–168 (1988).
8. H.-T. Lin and C.-C. Chen, Mixed convection from a vertical plate with uniform heat flux to fluids of any Prandtl number, *J. Chin. I. Ch. E.* **18,** 209–220 (1987).
9. H.-T. Lin, C.-C. Chen and W.-S. Yu, Mixed convection from a horizontal plate to fluids of any Prandtl number, *Wärme- und Stoffübertragung* **24,** 225–234 (1989).
10. H.-T. Lin, W.-S. Yu and C.-C. Chen, Comprehensive correlations for laminar mixed convection on vertical and horizontal flat plates, *Wärme- und Stoffübertragung* **25,** 353–359 (1990).